8. Appendix

8.1. Proof of Theorem 1

Proof. Let's denote

$$t_{d_n}^{(k-1)d_n+1} := \sqrt{d_n}(\Delta W_{d_n}^{(k-1)d_n+1} - \mu). \quad (16)$$

Then the variance estimator can be written as

$$\hat{\sigma}_n^2 = \frac{1}{K_n}\sum_{k=1}^{K_n}\left(t_{d_n}^{(k-1)d_n+1}\right)^2 - (t_n)^2 \quad (17)$$

where

$$t_n = \sum_{k=1}^{K_n} t_{d_n}^{(k-1)d_n+1}/K_n. \quad (18)$$

Note that $\mathbf{E}\left(t_{d_n}^{(k-1)d_n+1}\right) = \mathbf{E}(t_n) = 0$.

We will first show that the first term of the right hand side of (17) converges to σ^2 in L_2. Notice that

$$\mathbf{E}\left(t_{d_n}^{(k-1)d_n+1}\right)^2 = d_n\mathbf{E}\left(\Delta W_{d_n}^{(k-1)d_n+1} - \mu\right)^2 \quad (19)$$

$$\to \sigma^2 \quad (20)$$

as $n \to \infty$ and (thus) $d_n \to \infty$. Then it suffices to show

$$\mathbf{Var}\left(\frac{1}{K_n}\sum_{k=1}^{K_n}\left(t_{d_n}^{(k-1)d_n+1}\right)^2\right) \to 0. \quad (21)$$

This is true since

$$\mathbf{Var}\left(\sum_{k=1}^{K_n}\left(t_{d_n}^{(k-1)d_n+1}\right)^2\right) \le \sum_{k=1}^{K_n}\mathbf{E}\left(t_{d_n}^{(k-1)d_n+1)}\right)^4 \quad (22)$$

$$\le K_n C \quad (23)$$

when n and d_n are sufficiently large. Here the first less than or equal to sign follows from the assumption that for any two blocks $k < k'$, $\Delta W_{d_n}^{(k-1)d_n+1}$ and $\Delta W_{d_n}^{(k'-1)d_n+1}$ are uncorrelated, and the second less than or equal to sign follows from the assumption that $n^2\mathbf{E}(\Delta W_n - \mu)^4$ is uniformly bounded.

Now we only need to show that the second term of the right hand side of (17) converges to 0 in L_2, or equivalently, t_n converges to 0 in L_4.

Note that

$$t_n = \sqrt{d_n}(\Delta W_n - \mu) \quad (24)$$

and thus

$$\mathbf{E}(t_n^4) = d_n^2\mathbf{E}(\Delta W_n - \mu)^4 \to 0 \quad (25)$$

as $n \to \infty$ since $n^2\mathbf{E}(\Delta W_n - \mu)^4$ is bounded. □

8.2. Proof of Corollary 1.1

Proof. It suffices to show

$$\mathbf{Var}\left(\frac{1}{K_n}\sum_{k=1}^{K_n}\left(t_{d_n}^{(k-1)d_n+1}\right)^2\right) \to 0 \quad (26)$$

under the relaxed assumption. Consider for any $k < k'$

$$\mathbf{Cov}\left(\left(t_{d_n}^{(k-1)d_n+1}\right)^2, \left(t_{d_n}^{(k'-1)d_n+1}\right)^2\right) \quad (27)$$

$$\le \mathbf{E}\left(t_{d_n}^{(k-1)d_n+1}t_{d_n}^{(k'-1)d_n+1}\right)^2 \quad (28)$$

$$= d_n^2\mathbf{E}\left(\Delta W_{d_n}^{(k-1)d_n+1} - \mu\right)^2\left(\Delta W_{d_n}^{(k'-1)d_n+1} - \mu\right)^2 \quad (29)$$

$$= \frac{1}{d_n^2}\mathbf{E}\left(\sum_{\substack{i=(k-1)d_n\\+1}}^{kd_n}(Z_i - \mu)\right)^2\left(\sum_{\substack{i'=(k'-1)d_n\\+1}}^{kd_n}(Z_{i'} - \mu)\right)^2. \quad (30)$$

Then under the assumption that $\mathbf{E}(|(Z_i - \mu)(Z_j - \mu)(Z_{i'} - \mu)(Z_{j'} - \mu)|) \le \epsilon$ if n is sufficiently large for any i, j and i', j', we have

$$\mathbf{Var}\left(\frac{1}{K_n}\sum_{k=1}^{K_n}\left(t_{d_n}^{(k-1)d_n+1}\right)^2\right) \quad (31)$$

$$\le \frac{1}{K_n^2}\sum_{k=1}^{K_n}\mathbf{E}\left(t_{d_n}^{(k-1)d_n+1)}\right)^4 \quad (32)$$

$$+ \frac{2}{K_n^2}\sum_{1\le k<k'\le K_n}\mathbf{E}\left(t_{d_n}^{(k-1)d_n+1}t_{d_n}^{(k'-1)d_n+1}\right)^2 \quad (33)$$

$$\le \frac{1}{K_n^2}(K_nC + K_n^2\epsilon) = \frac{C}{K_n} + \epsilon. \quad (34)$$

which converges to 0 as $n \to \infty$. □